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Abstract: We review some methods for the numerical approximation of time dependent and steady first order Hamilton Jacobi equations. Most of the discussion focuses on conformal triangular type meshes but we show how to extend this to the most general meshes. We review some first order monotone schemes and also high order ones specially dedicated to steady problems

Key-words: Hamilton Jacobi equation, high order numerical schemes, unstructured meshes.

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Résumé : Nous présentons quelques méthodes numériques permettant l'approximation d'équations d'Hamilton Jacobi stationnaires ou non. La majeure partie de l'éxposé traite de ce qu' on peut faire pour des maillages non structurés de type triangulaire (ou tétrahédriques), mais on montre comment traiter les cas les plus généraux. On présente des schémas monotones du premier ordre mais aussi des schémas d' ordre élevé spécialement adaptés au cas stationnaire.

Mots-clés : Equation d'Hamilton Jacobi du premier ordre, schémas numériques d'ordre élevé, maillages non structurés.

1 Introduction

This paper describes some of the schemes that are currently used to compute approximated solutions of first order Hamilton Jacobi equations, namely the steady Dirichlet problem

$$\begin{array}{ll} H(x,u,Du)=0 & x\in\Omega\subset\mathbb{R}^d \\ u=\varphi & x\in\partial\Omega \end{array}$$

or the Cauchy problem

$$\begin{array}{l} \dfrac{\partial u}{\partial t}+H(x,u(x),Du)=0 \quad x\in\mathbb{R}^d, t>0 \\ u(x,0)=u_0(x). \end{array}$$

In many problems of physical interest, one needs to compute the solution of such an equation. One of the simplest examples is the computation of a distance function, more sophisticated examples consist in evaluating the arrival time of a front (wave front, flame front, etc) in a non homogeneous media. Similar problem also exist in control theory, thermodynamics, etc.

We will concentrate on the numerical approximation of these equations on conformal triangular type meshes. This is a more general situation than the standard Cartesian meshes where this problem was considered originally, but still less general than the case of non conformal meshes. However, we sketch how to extend the schemes we describe here to the most general case. Indeed the situation we consider is an intermediate one, it is general enough to be obliged to imagine solutions that are generic enough and not too specific so that the structure of the mesh does not play a too rigid role.

In the first part, we quickly recall the notion of viscosity solution for HJ equations, either for the Cauchy problem or for the steady one with Dirichlet boundary conditions. Then, in two particular cases we recall the exact solution. The next section is devoted to the numerical approximation of the Cauchy problem. In the third section we discuss the approximation of the Dirichlet problem. The fourth section considers a general formulation for high order discretisation. The bridge between Cartesian meshes and non conformal meshes is sketched in the fifth section. The last section is devoted to some numerical applications.

As we have already said, our point of view is quite biased. There are basically two classes of approximation techniques. The first one tries to directly use the notion of viscosity solution, see section 2, this our point of view. In the second class of methods, one tries to exploit the formal link between some systems of conservation laws and the HJ equations. The link is that if one differentiates the equation

$$\frac{\partial u}{\partial t}+H(Du)=0$$

with respect to x and y, denoting $p_i=\dfrac{\partial u}{\partial x_i}$ and $p=Du$, we have

$$\frac{\partial p_i}{\partial t}+\frac{\partial H}{\partial x}(p)=0.$$

This is the point of view of the papers that extend finite volume or Discontinuous Galerkin methods, see [1] for an example.

2 Short review on the HJ equations and viscosity solutions

We consider the Cauchy problem : find $u\in C^0(\Omega)$, the space of continuous function on the open subset $\Omega\subset\mathbb{R}^d$, such that

$$\begin{array}{ll} H(x,u,Du)=0 & x\in\Omega\subset\mathbb{R}^d \\ u=g & x\in\partial\Omega \end{array} \tag{1}$$

in the viscosity sense. In (1), $(x,s,p)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d\mapsto H(x,s,p)$ is uniformly continuous.

Before going further, let us briefly review the notion of viscosity solution for (1). For any function z, we consider the upper semi–continuous (u.s.c) and lower semi–continuous (l.s.c) envelopes of z with respect to all the variables. They are defined by

$$z^*(x) = \limsup_{x \to y} z(y) \text{ and } z_*(x) = \liminf_{x \to y} z(y).$$

Following [2], we introduce the function G

$$G(x,s,p) = \begin{cases} H(x,s,p) & x \in \Omega \\ s - g(x) & x \in \partial\Omega. \end{cases}$$

The computation of G_* and G^* is easy and we have:

$$\begin{cases} G_*(x,s,p) = G^*(x,s,p) = H(x,s,p) & \text{if } x \in \Omega \\ G_*(x,s,p) = \min(H(x,s,p), s - g(x)) & \text{if } x \in \partial\Omega \\ G^*(x,s,p) = \max(H(x,s,p), s - g(x)) & \text{if } x \in \partial\Omega \end{cases} \tag{2}$$

A locally bounded u.s.c function u defined on $\overline{\Omega}$ is a viscosity sub-solution of (1) if and only if, for any $\phi \in C^1(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local maximum of $u - \phi$, then

$$G_*(x_0, u(x_0), D\phi(x_0)) \leq 0. \tag{3}$$

Similarly, u, a locally bounded, l.s.c. function defined on $\overline{\Omega}$ is a viscosity super-solution of (1) if and only if, for any $\phi \in C^1(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local minimum of $u - \phi$, then

$$G^*(x_0, u(x_0), D\phi(x_0)) \geq 0. \tag{4}$$

A viscosity solution is simultaneously a sub– and a super-solution of (1). This can be generalized to other types of boundary conditions such as Neumann, etc.

In the case of the Cauchy problem,

$$\begin{array}{l} \dfrac{\partial u}{\partial t} + H(x, u(x), Du) = 0 \quad x \in \mathbb{R}^d, t > 0 \\ u(x,0) = u_0(x) \end{array} \tag{5}$$

where u_0 belongs to the set of bounded and uniformly continuous functions, $BUC(\mathbb{R}^2)$. One can adapt easily the arguments raised for the steady problem. Here, G is simply

$$G(x,s,p) = p_t + H(x,s,p_x), x \in \mathbb{R}^d, s \in \mathbb{R}^d, p = (p_t, p_x) \in \mathbb{R} \times \mathbb{R}^d$$

so that $G_* = G^* = G$. Subsolutions (resp. supersolutions) are elements of $BUC(\mathbb{R}^d \times [0,T])$ where $T > 0$, so that inequality (3) (resp. (4)) holds.

All this can be extended to the Cauchy–Dirichlet problem

$$\begin{array}{ll} \dfrac{\partial u}{\partial t} + H(x, u(x), Du) = 0 & x \in \Omega \subset \mathbb{R}^d, t > 0 \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = g(x,t) & x \in \partial\Omega, t > 0 \end{array} \tag{6}$$

Under standard assumptions on the open subset Ω, g and H and u_0, one can prove existence and uniqueness of the viscosity solutions of (1), (5) and (6), see [2]. In particular, this is true if the Hamiltonian H is convex in $p \in \mathbb{R}^d$ and if $\partial\Omega$ lipschitz continuous.

In this paper, we assume that (1) has a uniqueness principle, that is any sub–solution u and any super–solution v of (1) satisfy

$$\forall x \in \Omega, \qquad u(x) \leq v(x) \tag{7}$$

and

$$\forall x \in \mathbb{R}^d, t > 0, \qquad u(x,t) \leq v(x,t) \tag{8}$$

in the case of the Cauchy problem.

3 Some exact solutions

Two examples are considered. Either we look for the steady problem (1) with a convex Hamiltonian, or we look for the Cauchy problem (5) with either a convex (or concave Hamiltonian) or a convex (or concave) initial condition.

The main tool is the Legendre transform. If f is a convex function such that

$$\lim_{||x||\to+\infty} \frac{f(x)}{||x||} = +\infty \tag{9}$$

we define the Legendre transform of f by

$$f^\star(p) = \sup_{y\in\mathbb{R}^d} \big(p\cdot y - f(y)\big).$$

If the supremum is reached at $y^\star$, we have the relation

$$f^\star(p) + f(y^\star) = p\cdot y^\star.$$

This shows that $f^\star(p)$ can be seen as the abscissa of the tangent of the graph of f at y^*. This graphic interpretation helps to see that, if f is regular enough, the graph of f is the envelope of its tangent, so that

$$(f^\star)^\star = f.$$

Of course this relation is generalizable to a convex f when it satisfies (9).

All this generalizes to a concave functions (since $-f$ is convex),

$$f^\star(p) = -(-f)^\star(p) = \inf_{y\in\mathbb{R}^d} \big(-y\cdot p - f(y)\big).$$

3.1 For the steady problem

We assume that the Hamiltonian is given by

$$H(x,u,p) = \sup_{v\in V} \{-b(x,v)\,.\,p + \lambda u - f(x,v)\}$$

where the space of controls V is compact, and we have standard assumptions on b, f and $\lambda > 0$, see [2]. For the Dirichlet condition, the solution of (1) is given by the dynamical programming principle, for any $T > 0$,

$$\begin{aligned} u(x) = \inf_{v(.)} \Bigg[\int_0^{\min(T,\tau)} & f(y_x(t),v(t))e^{-\lambda t}dt \\ & + 1_{\{T<\tau\}} u(y_x(T))e^{-\lambda T} + 1_{\{T\geq\tau\}}\varphi(y_x(\tau))e^{-\lambda\tau}\Bigg] \end{aligned} \tag{10}$$

The trajectory $y_x(.)$ satisfies $y_x(0) = x \in \Omega$ and

$$\frac{d}{dt}y_x(t) = b(y_x(t),v(t)) \text{ for } t > 0.$$

They are defined if f is regular enough, say Lipschitz continuous. The exit time τ is

$$\tau = \inf\{t \geq 0, y_x(t) \notin \Omega\}.$$

Details can be found in [3, 2]

The analytical expression for the solution of (5), when H only depends on $p \in \mathbb{R}^d$, is given in [4],

1. when H is uniformly Lipschitz continuous and u_0 convex,

$$u(x,t) = \sup_{p\in\mathbb{R}^d} \Big[x\cdot p - u_0^\star(p) - tH(p)\Big], \tag{11}$$

when u_0 is concave,

$$u(x,t) = \inf_{p\in\mathbb{R}^d} \Big[-x\cdot p + u_0^\star(p) - tH(p)\Big]. \tag{12}$$

2. when u_0 is uniformly continuous, we have for a convex Hamiltonian

$$u(x,t) = \inf_{p\in\mathbb{R}^d} \left[u_0(y) + tH^\star\left(\frac{y-x}{t}\right)\right], \tag{13}$$

and for a concave Hamiltonian

$$u(x,t) = \sup_{p\in\mathbb{R}^d} \left[u_0(y) + tH^\star\left(\frac{x-y}{t}\right)\right]. \tag{14}$$

The formula (11) and (12) reflect the Huygens's principle, while (13) and (14) are consequence of the dynamical programing principle (10).

Note that if u_0 is linear in x, $u_0(x) = A + p\cdot x$, we have

$$u(x,t) = u_0(x) - tH(p).$$

These results are only valid for special initial conditions or particular Hamiltonian. We have the more general results

Lemma 1 (Bardi-Osher[5]). *If $u_0 = u_0^{conv} + u_0^{conc}$ where u_0^{conv} (resp. u_0^{conc}) is convex (resp. concave), then the solution u of (5) satisfies*

$$\forall (x,t)\in\mathbb{R}^d\times[0,T], \qquad \psi_2(x,t)\le u(x,t)\le \psi_1(x,t)$$

with

$$\psi_1(x,t) = \inf_{q\in\mathbb{R}^d}\sup_{p\in\mathbb{R}^d}\left(x\cdot p - \left(u_0^{conv}\right)^\star(p) - \left(u_0^{conc}\right)^\star(q) - tH(p-q)\right)$$

and

$$\psi_2(x,t) = \sup_{p\in\mathbb{R}^d}\inf_{q\in\mathbb{R}^d}\left(-x\cdot q - \left(u_0^{conv}\right)^\star(p) - \left(u_0^{conc}\right)^\star(q) - tH(q-p)\right)$$

Proof. The proof uses the fact that $u_0^{conc}(x) = \sup_{p\in\mathbb{R}^d}\left(x\cdot p - \left(u_0^{conc}\right)^\star(p)\right)$ so that for any p,

$$u_0^{conc}(x) \ge v_{p,0}(x) := x\cdot p - \left(u_0^{conc}\right)^\star(p).$$

Then we solve the Cauchy problem for $v_0 + u_0$ which is convex, use the comparison principle (8), and take the maximum. This gives the first inequality. The second one is obtained in a similar way. □

Lemma 2 ([6]). *If $H = H_{conv} + H_{conc}$ where H_{conv} (resp. H_{conc} is convex (resp. concave) uniformly continuous, the solution of the Cauchy problem (5) satisfies*

$$\Phi_2(x,t)\le u(x,t)\le \Phi_1(x,t)$$

with

$$\Phi_1(x,t) = \inf_{q\in\mathbb{R}^d}\sup_{y\in\mathbb{R}^d}\left[u_0(y) + tH^\star_{conv}(q) + tH^\star_{conc}\left(\frac{y-x}{t}+q\right)\right]$$
$$\Phi_2(x,t) = \sup_{p\in\mathbb{R}^d}\inf_{y\in\mathbb{R}^d}\left[u_0(y) + tH^\star_{conc}(q) + tH^\star_{conv}\left(\frac{x-y}{t}+q\right)\right]$$

The proof is similar and is given in [6].

4 First order approximation of the Cauchy problem

In order to simplify the text, we assume from now on $d = 2$, but all the results can be easily generalized to other dimensions, in particular $d = 3$. We consider a triangulation of $\mathbb{R}^2$, the vertices are $\{M_i\}_{i=1,n_s}$, the triangles are $\{T_j\}_{j=1,n_t}$. We denote by T a generic triangle. The vertices of T_k are M_{i_1}, M_{i_2} and M_{i_3}, for simplicity we often denote them by i_1, i_2, i_3 or by 1, 2, 3 when there is no ambiguity. The family of triangulations we consider is shape regular.

Up to our knowledge, the first paper to discuss in detail the approximation of (5) is [7]. As in this reference, (5) is approximated by

$$\begin{array}{l} u_i^{n+1} = u_i^n - \Delta t\mathcal{H}_i, \quad i = 1,\cdots n_s, n\in\mathbb{N}^* \\ u_i^0 = u_0(M_i) \end{array} \tag{15}$$

where $\Delta t > 0$ is the time step and u_i^n is an approximation of $u(M_i, n\Delta t)$, and the numerical Hamiltonian $\mathcal{H}_i$ depends on u_i^n , the values of u_j^n where $j \in \mathcal{V}_i$ ($\mathcal{V}_i$ is the set of neighbors of M_i including M_i by convention), and if necessary on M_i,

$$\mathcal{H}_i := \mathcal{H}(M_i, u_i^n, \{u_i^n\}_{j\in\mathcal{V}_i}). \tag{16}$$

In this reference is introduced the notion of consistency. The numerical Hamiltonian $\mathcal{H}_i$ is consistant when, if $v_i = A + p\cdot\vec{OM_i}$, then what ever M_j and $s\in\mathbb{R}$,

$$\mathcal{H}(M_j, s, \{v_i\}_{j\in\mathcal{V}_i}) = H(M, s, p). \tag{17}$$

A less restrictive definition, which is helpful for the proof, is given in [8],

Definition 1. *We say that the Hamiltonian $\mathcal{H}$ is weakly consistent if for all $x\in\overline{\Omega}$ and $\phi\in C_b^\infty(\overline{\Omega})$ (the set of C^∞ bounded functions),*

$$\limsup_{h\to 0, y\to x,\xi\to 0}\mathcal{H}(y,\phi(y)+\xi,\phi+\xi) \le G^*(x,\phi(x),D\phi(x)) \tag{18}$$

and

$$\liminf_{\rho\to 0, y\to x,\xi\to 0}\mathcal{H}(y,\phi(y)+\xi,\phi+\xi) \ge G_*(x,\phi(x),D\phi(x)). \tag{19}$$

A scheme that satisfies (17), $\mathcal{H}$ is strongly consistant. A strongly consistent scheme is weakly consistent.

The structure of the solution of (5) was first used in [5, 1]. In particular, they used the results of Lemma 1 to define a Godunov–like scheme. In [6], it is shown that, in general, for non structured meshes, the generalization of the Godunov–like scheme of [5] leads to a non consistant Hamiltonian. However, Lemma 2 provides a solution.

Assume that $\{u_j^n\}_{j=1,\cdots,n_s}$ is known and denote by u_h^n the piecewise linear interpolation of this data. For any mesh point M_i, we consider set $\{\Omega_i\}_{i=1,\cdots,\omega_i}$ of angular sectors at M_i, see figure 1. Each angular sector Ω_i corresponds to one of the triangles that share M_i and we denote by U_i the gradient of u_h^n in that triangle. The functions Φ_1 and Φ_2 evaluated at $x = M_i$ are

$$\Phi_1(M_i,\Delta t) = u_i^n - \Delta t\min_{q\in\mathbb{R}^2}\max_{1\le k\le\omega_i}\sup_{z\in\Omega_k}\left(U_i\cdot z - H_1^*(z-q) - H_2^*(q)\right)$$
$$\Phi_2(M_i,\Delta t) = u_i^n - \Delta t\max_{q\in\mathbb{R}^2}\min_{1\le k\le\omega_i}\inf_{z\in\Omega_k}\left(U_i\cdot z - H_1^*(z-q) - H_2^*(q)\right)$$

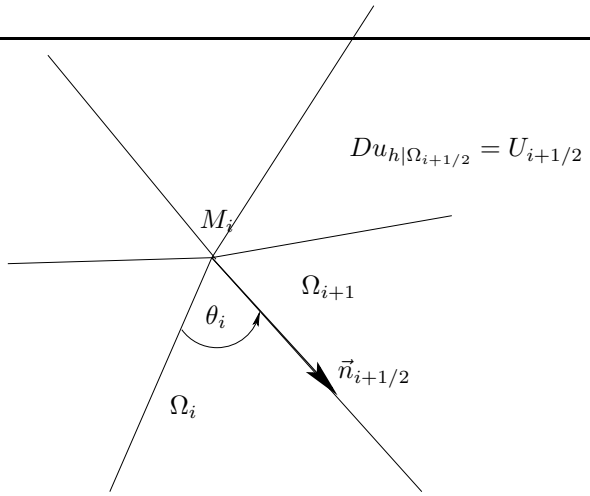


Figure 1: Illustration of the angular sectors Ω_i, θ_i and the vectors $\vec{n}_{i+1/2}$ that are needed in numerical Hamiltonians definitions.

The key remark is that any of the terms multiplied by Δt, say $\min_{q\in\mathbb{R}^2}\max_{1\le k\le\omega_k}\sup_{z\in\Omega_k}\Big(U_i\cdot z - H_1^*(z-q) - H_2^*(q)\Big)$, reduces to $H(p$ when $U_i \equiv p\ \forall i$. Hence, any of these terms defines a consistant numerical Hamiltonian, for example,

$$\mathcal{H}_i := \max_{q\in\mathbb{R}^2}\min_{1\le k\le\omega_i}\inf_{z\in\Omega_k}\big(U_i\cdot z - H_1^*(z-q) - H_2^*(q)\big), \tag{20}$$

the dependency in u_j^n, $j\in\mathcal{V}_i$ appears in the gradients U_i. This formula can easily be extended to the more general case $H = H(x,u,Du)$ and simplifies when H is convex, for example

$$\mathcal{H}_i = \max_{1\le k\le\omega_i}\max_{z\in\Omega_i}\big(U_i\cdot z - H^*(z)\big). \tag{21}$$

A second remark is that, by construction, $\mathcal{H}_i$ defined by (20) or (21) is monotone, that is

Definition 2 (Monotone Hamiltonians). *We say that $\mathcal{H}$ is monotone if, whatever $M_i\in\Sigma$, $u_j\le v_j$, and for any $s\in\mathbb{R}$,*

$$\mathcal{H}(M_i,s,\{u_j\}_{j\in\mathcal{V}_i}) \ge \mathcal{H}(M_i,s,\{v_j\}_{j\in\mathcal{V}_i}).$$

The Hamiltonian (20) is monotone by construction if $\Delta t/h\max_{||Du_{|T}||_\infty\le L}||D_pH(p)||_\infty \le 1/2$: this is a consequence of (8) and the inequalities

Another key remark is that the value of $\mathcal{H}_i$ defined by (20) or (21) does not depend on the structure of the mesh, but on the interpolant u^n. In other words, if one splits an angular sector Ω_k in two, *without changing the value of* U_k, the numerical Hamiltonian is not modified. We say that the scheme is *intrinsic* and we have the following error estimate,

Theorem 1 ([6]). *Let $H : \mathbb{R}^2\to\mathbb{R}^2$ be continuous and $u_0\in BUC(\mathbb{R}^2)$ and Lipschitz continuous (with constant L). Let $\mathcal{T}$ be a triangulation where h is the largest radius of the circles of center M_i, $i=1,\cdots,n_s$ and contained in all the triangles having M_i as a vertex.. We assume the mesh shape regular, i.e. the minimum angle α of the triangles T is uniformly bounded from bellow.*

Let u be the viscosity solution of (5) and $\{u_i^n\}_{j=1,\cdots,n_s}$ be defined by (15). Then there exists a constant c which depend only on α, L, $T>0$ and $\mathcal{H}$ such that for any M_i and n with $0\le n\Delta t\le T$,

$$\Big|u_i^n - u(M_i,n\Delta t)\Big| \le c\sqrt{\Delta t}$$

The proof is an adaptation of the main result of [7] with some technicalities (in particular for showing that the time step only depend on u_0) specific to unstructured meshes. We refer to [6].

The practical evaluation of the Legendre transform is not always an easy task, so other numerical Hamiltonians exists. The simplest one is the Lax Friedrichs one, which is inspired by the Lax Friedrichs scheme for conservation laws. It has several versions. The first one is

$$\mathcal{H}_i^{LF}(Du_{h|\Omega_1},\cdots,Du_{h|\Omega_{k_i}}) = H(\bar{U}) - \frac{\epsilon}{h}\oint_{C_h}[u_h(M)-u_h(M_i)]dl, \tag{22a}$$

where C_h (resp. D_h) is a circle (disk) of center M_i and radius h,

$$\widehat{U} = \frac{\int_{D_h} Du_h\, dxdy}{\pi h^2},$$

and ϵ is larger than any Lipschitz constant of H divided by 2π. This defines a monotone scheme provided that $\Delta t/h \leq \frac{\varepsilon}{2\pi}$.

A different version of the Lax Friedrichs Hamiltonian, that is monotone under the same constraint, is the following :

$$\mathcal{H}_h^{LF}(Du_{h|\Omega_1},\cdots,Du_{h|\Omega_{k_i}}) = \frac{\displaystyle\int_{D_h} H(Du_h)}{\pi h^2} - \frac{\epsilon}{h}\oint_{C_h}[u_h(M)-u_h(M_i)]dl. \tag{22b}$$

This version can be rewritten as

$$\mathcal{H}_h^{LF}(Du_{h|\Omega_1},\cdots,Du_{h|\Omega_{k_i}}) = \sum_{0\leq l\leq k_i}\frac{\theta_l^i}{2\pi}H(Du_{h|\Omega_l^i}) + \varepsilon\sum_{0\leq l\leq k_i}\tan\theta_l^i\frac{\vec{n}_{l-1/2}^i+\vec{n}_{l+1/2}^i}{2}\cdot Du_{h|\Omega_l^i}.$$

The vector $\vec{n}_{l+1/2}$ is the unit vector of the edge that separates the angular sectors Ω_l and Ω_{l+1}, the angle θ_l^i is the angle of the angular sector at M_i, see Figure 1. The parameter ε is the same as in the previous version.

A third version is

$$\mathcal{H}^{LF}(Du_{h|\Omega_1},\cdots,Du_{h|\Omega_{k_i}}) = \frac{\displaystyle\sum_{T\ni M_i}|T|H(Du_{h|T}) + \alpha\sum_{M_j\in T}(u_i-u_j)}{\displaystyle\sum_{T\ni M_i}|T|} \tag{22c}$$

and $\alpha \geq h_T \max_p ||D_pH||$ where h_T is the largest edge of T.

The main difference between these different formulas is that (22a) and (22b) are intrinsic in the sense given in [6] while (22c) is not. Hence, following the same reference, (22a) and (22b) are convergent and the error estimate is $\mathcal{O}(h^{1/2})$. For (22c), such an error estimate is not available (at least when following the technique of [6], but it is convergent : this is a simple application of [8].

The advantage of (22c) over the other two versions is its simplicity in coding compared to (22a) and (22b).

5 The Dirichlet problem

The approximation of the Dirichlet problem is not as simple as it looks. An illustration is to find $u : [0,1] \to \mathbb{R}$ such that

$$|u'| - 1 = 0 \text{ in } x \in [0,1], \qquad u(0) = 1 \text{ and } u(1) = 2$$

which has no classical solution, but which viscosity solution, defined only in $[0,1[$ is $u(x) = x$. We have $\lim_{x\to 1^-} u(x) = 1 \neq 2$. In other cases, say $u(0) = u(1) = 0$, we have $u(x) = |x-1/2|$ which matches strongly the boundary conditions.

In order to define a scheme, we start from (10), and consider a triangulation of Ω . First we assume that $M_i \in \partial\Omega$. In (10), the set of controls can be splitted into two parts: the set V_1 for which $T < \tau$, and V_2 for which $T \geq \tau$. Hence,

$$u(x) = \min(\inf_{v\in V_1}[\cdots], \inf_{v\in V_2}[\cdots]).$$

Let $\vec{n}$ be the interior normal to Ω at $x \in \overline{\Omega}$. Since T is arbitrary, it can be chosen as small as possible. In the limit $T \to 0$, the set V_1 would be the set of controls for which $b(x,v)\,.\,\vec{n} > 0$, i.e. the control for which the trajectory goes into Ω. The dynamical programing principle $\inf_{v\in V_1}[\cdots] - u(x) = 0$ corresponds to the Hamiltonian

$$H_b(x,t,p) = \sup_{v\in V_1}\{b(x,v)\,.\,p + \lambda t - f(x,v)\}.$$

We also have the relation $H_b \leq H$.

The "inf" on V_2 can be approximated, if T is small, by $\varphi(y_x(\tau))$. Since $T \leq \tau$, and if we can choose controls for which $T \simeq \tau$, we get

$$\varphi(y_x(\tau)) \simeq \varphi(x)$$

because φ is continuous. We see that (10) can be approximated, at a boundary point, by

$$0 = \max(\mathcal{H}_i^b, u(x) - \varphi(x))$$

where $\mathcal{H}^b$ is a consistant approximation of H_b.

When $M_i \notin \partial\Omega$, taking T small enough, we can see formally that the boundary plays no role so that we can take any consistant Hamiltonian, for example those defined in the previous section.

The scheme is then

$$\mathcal{S}(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}) = 0 \qquad \forall i \tag{23}$$

with

$$\mathcal{S}(x,s,\{u_j\}_{j\in\mathcal{V}_i}) = \begin{cases} \mathcal{H}(x,s,\{u_j\}_{j\in\mathcal{V}_i}) & \text{if } x \notin \partial\Omega \\ \max(\mathcal{H}_b(x,s,\{u_j\}_{j\in\mathcal{V}_i}), s-\varphi(x)) & \text{else.} \end{cases} \tag{24}$$

The scheme (23)–(24) cab be extended to other types of boundary conditions. There is an implicit dependency of $\mathcal{S}$ with respect to h. We extend the definition of $\mathcal{S}$ to any $y \in \Omega$ by saying that $\mathcal{S}(x,s,\{u_j\}_{j\in\mathcal{V}_j}) = \mathcal{S}(M_i,s,\{u_j\}_{j\in\mathcal{V}_j})$ if x belongs to the dual control volume associated to M_i.

We have the following result

Theorem 2 ([9]). *Assume that*

1. $H_b \leq H$,
2. $\mathcal{H}$, $\mathcal{H}^b$ *are monotone and the solutions of* (23) *are uniformly bounded,*
3. *for all* $\phi \in C_b^\infty(\overline{\Omega})$, *we have*

$$\textit{For any } x \in \overline{\Omega},\quad \lim_{h\to 0, y\to x, \xi\to 0} \mathcal{H}(y,\varphi(y)+\xi,\varphi+\xi) = H(x,\varphi(x),D\varphi(x)) \tag{25a}$$

$$\textit{For any } x \textit{ in a neighborhood of } \partial\Omega,\ \lim_{h\to 0, y\to x, \xi\to 0} \mathcal{H}^b(y,\varphi(y)+\xi,\varphi+\xi) = H_b(x,\varphi(x),D\varphi(x)) \tag{25b}$$

4. *The equation* (1) *has a uniqueness principle,*

Then the family u_h *defined by* (24) *converges locally uniformly to the solution of* (1) *in* Ω.

Proof. The key argument of the proof is a convergence result by Barles and Souganidis [8]. □

Unfortunately, this results is not enough to guaranty a "good" convergence. Take the example at the beginning of the section, a regular mesh $(1/\Delta x = N+1)$, the Godunov scheme that reduces here to

$$\mathcal{H}_i = \mathcal{H}(u_{i-1}, u_i, u_{i+1}) = \max\left(\frac{|u_{i+1}-u_i|}{\Delta x}, \frac{|u_{i-1}-u_i|}{\Delta x}\right) - 1$$

and $\mathcal{H}_b = -\infty$. This amounts to setting $u_0 = 0$, $u_N = 2$. Theorem 2 applies but numerical experiments indicate that the gradient of the solution is not bounded so that there is no hope to have a convergence like Δx^α with $\alpha > 0$ reasonable.

In [9], this problem is studied and it is shown that if H is convex, if the Godunov scheme constructed on the boundary Hamiltonian H_b is constructed, and if a coercivity assumption holds for H, H_b and the associated numerical Hamiltonian, then one can control Du, and it is shown in [10] that the error behaves like $h^{-1/2}$. Similar error estimates (for Cartesian meshes) were obtained in [11].

6 High order extension

Up to now, all the examples we have given are only first order accurate schemes. There are several ways constructing high order schemes.

One possible construction is a consequence of the following fact. The "Du" dependency in the Hamiltonian comes form the term "$\{u_j, j \in \mathcal{V}_i\}$" in (16). More precisely, in all the known examples, this dependency occurs through differences, $u_j - u_i$ for $j \in \mathcal{V}_j$. These terms can be rewritten in terms of the gradients of u in the triangles surrounding M_i (this remark has already been used in (22a), (22b) and (22c)). One can exploit this remark, as in [1] for example, by modifying the evaluation of the gradients in the triangles. Instead of linear interpolant, one can use higher degree polynomials thanks to the ENO/WENO methodology, [6, 12, 13].

A other solution is the Discontinuous Galerkin strategy [14, 15, 16]. We do not detail this technique here.

A last method is a blending strategy, [17]. The idea is to blend a low order, monotone Hamiltonian $(\mathcal{H}^M)$ with a high order Hamiltonian consistent Hamiltonian $(\mathcal{H}^H)$. By high order we mean that if u is a smooth solution of (1), then

$$\mathcal{H}^H(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}) = O(h^k) \tag{26}$$

for $k > 1$. The scheme writes

$$\mathcal{H}(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}) = \ell_i \mathcal{H}^M(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}) + (1-\ell_i)\mathcal{H}^H(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}) + \varepsilon(h) \tag{27}$$

where $\varepsilon(h) = Ch^k$ for some positive constant C and ℓ_i is chosen such that, if $r_i := \frac{\mathcal{H}_i^H}{\mathcal{H}_i^M}$, we have

$$\ell_i + (1-\ell_i)r_i \geq \varepsilon'(h). \tag{28}$$

where $\varepsilon'(h)^{-1}\varepsilon(h) = o(1)$. We have the simple lemma which proof is immediate,

Lemma 3. *If $\mathcal{H}^M$ and $\mathcal{H}^H$ are strongly consistent, $\mathcal{H}$ defined by (27) is weakly consistent.*

The justification of (28) comes from the simple relation

$$\mathcal{H}(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}) = \Big(\ell_i + (1-\ell_i)r_i\Big)\mathcal{H}^M(M_i, u_i\{u_j\}_{j\in\mathcal{V}_i}) + \varepsilon(h) \tag{29}$$

from which, using once more the technique of the convergence result of [8], one can show for the scheme (23)-(24) where $\mathcal{H}$ is given by (27) the following result

Theorem 3. *[17]] We consider the scheme (23) where, in (24) $\mathcal{H}$ is defined by (27). We assume that*

1. *$\mathcal{H}^M$, $\mathcal{H}^H$ and $\mathcal{H}_b$ are strongly consistent*
2. *$\mathcal{H}^M$ and $\mathcal{H}_b$ are monotone Hamiltonians,*

4. *The blending parameter ℓ belongs to $[0,1]$ and satisfies*

$$r = \frac{\mathcal{H}^H(x, u_h(x), u_h)}{\mathcal{H}^M(x, u_h(x), u_h)}, \qquad \ell(x) + (1 - \ell(x))r \geq \varepsilon'(h)$$

where the parameters $\varepsilon(h)$ and $\varepsilon'(h)$ satisfies $\varepsilon'(h)^{-1}\varepsilon(h) = o(1)$,

5. *There exists a unique solution u_h of (23) that satisfies L^∞ bound that is uniform in h,*

6. *The equation (1) has a uniqueness principle.*

Then the family u_h defined by the scheme converges locally uniformly to the solution of (1) in Ω.

Examples of the blending parameter are, given constants $\alpha_- \geq 1$, $\alpha_+ > 0$ and $\beta > 0$, ℓ^*

$$\ell = \begin{cases} \min(1, \alpha_- |r|) & \text{if } r \leq 0 \\ 0 & \text{if } 0 \leq r \leq \beta \\ \min(1, \alpha_+(r - \beta)) & \text{else.} \end{cases} \tag{30}$$

This the one we have chosen in practical applications with $\beta = 0$ and $\alpha_+ = 1$. Implementation details can be found in [17].

7 Links between Cartesian and non conformal meshes

It is not difficult to construct numerical Hamiltonians that work on general non conformal meshes. The only key point is to construct *monotone* Hamiltonians. The convergence results of [8] and Theorem 2 can easily be adapted : a close inspection of the proof shows that the structure of the mesh plays *no* role. What matters is to define, for any mesh point M_i, a local interpolation, π that operates on $U_i := \{u_j\}_{j \in \mathcal{V}_j}$ onto the space of piecewise linear functions, and such that if $u_j \leq v_j$, $j \in \mathcal{V}_j$, then $\pi(U_i) \leq \pi(V_i)$.

Consider Figure 2.The neighbors of M_i are $\{P_i\}_{i=1,\cdots,8}$, from which we construct a local triangulation (dotted lines) that is used to define a piece–wise linear interpolant. It does not need to be continuous. Then we can use our Hamiltonians to define schemes that are clearly consistant, monotone. The tricky part is the choice of the neighbors. Figure 2 shows an extreme case. A probably better choice would have been to choose only $\{P_2, P_3, P_4, P_5, P_7\}$ because the aspect ratio of the triangles is larger.

Note that the Hamiltonians of [1], thanks to this set of remarks, are particular cases of our formula.

8 Numerical results

In general, it is difficult to compute analytically the solution of a first order Hamilton–Jacobi equation, and the situation is even worse when the Hamiltonian is not convex (nor concave) because the analogy with hyperbolic systems becomes looser in general. Hence, it becomes more difficult to judge the quality of numerical results. To overcome this difficulty in a special case, we consider $H(p) = (||p|| - 1)^3$ and the problem

$$\begin{array}{ll} H(Du) = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ u = 10 & \text{on } \Gamma_2 \end{array} \tag{31}$$

where Ω is depicted in figure 3. Since $t \mapsto t^3$ is monotone increasing, u is solution of (31) if and only if it is a solution of

$$\begin{array}{ll} ||Dv|| - 1 = 0 & \text{on } \Omega, \\ v = 0 & \text{on } \Gamma_1, \\ v = 10 & \text{on } \Gamma_2. \end{array} \tag{32}$$

The solution of (31) and (32) is the distance to Γ_1.

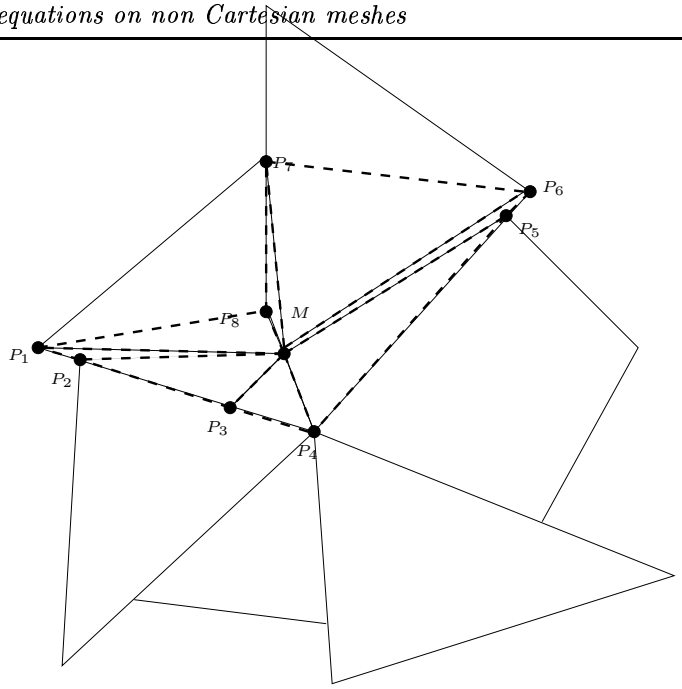


Figure 2: Case of a non conformal mesh.

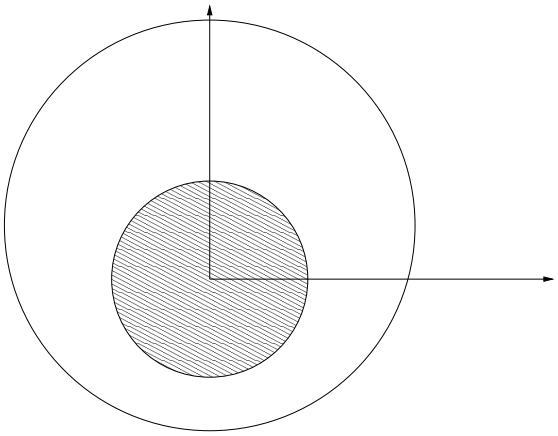
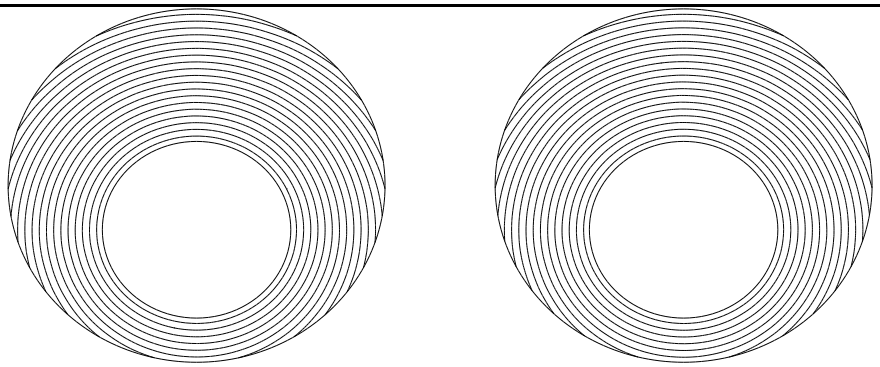


Figure 3: Computational domain for problem (31. Γ_1 is the inner circle of center $(0,0)$ and radius $r=1$, Γ_2 is the outer circle (center $(0,0.5)$, radius $r=3$).

In order to discretize (31), we write $H = H_1 + H_2$ with $H_1(p) = \max(|p|| - 1, 0)^3$ and $H_2(p) = \min(||p|| - 1, 0)^3$. These functions are respectively convex and concave. The numerical Hamiltonian is the Lax Friedrich's and the boundary Hamiltonian is Godunov's. The numerical solution is displayed on Figure 4-(a). The solution of (32) with the Godunov Hamiltonian is provided on Figure 4-(b). A close comparison show that they are (almost) identical.



(a) (b)

Figure 4: (a) : Solution of problem (31), min=0, max=1.48. (b) : Solution of problem (32), min=0, max=1.504

An other application of the boundary conditions is given by the approximation of the following problem, on the same geometry,

$$\begin{array}{ll} H(Du) = 0 & \text{on } \Omega, \\ u(x,y) = 0 & (x,y) \in \Gamma_1, \\ u(x,y) = 3\cos(2\pi x) & (x,y) \in \Gamma_2. \end{array} \tag{33}$$

Since H is non convex, it is difficult to know *a priori* what would be the value of the solution on the boundary. The computed solution is given on Figure 5-(a). It can be seen that the solution satisfies strongly the boundary condition on Γ_2 and only weakly on Γ_1 (contrarily to the previous example). Note however that they have been numerically *weakly* imposed on Γ_1 and Γ_2. The solution is also in very good agreement with the one obtained from the discretization of

$$\begin{array}{ll} ||Dv|| - 1 = 0 & \text{on } \Omega, \\ v(x,y) = 0 & (x,y) \in \Gamma_1, \\ v(x,y) = 3\cos(2\pi x) & (x,y) \in \Gamma_2. \end{array} \tag{34}$$

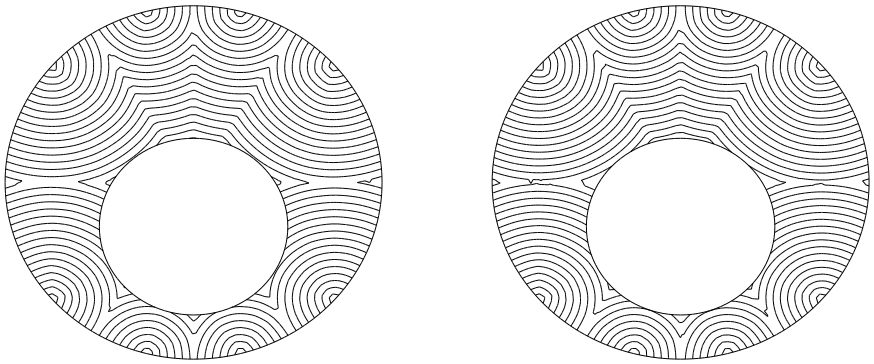
which is displayed on Figure 5-(b).

We also show how the high order extension of section 6 works when the Godunov solver with P^2 interpolation for the high order scheme. The zoom is displayed on Figure 6. Clearly, a very large overshoot exists where u is not C^1, there the solution of the blended scheme is monotone and is very similar to the first order one. In the smooth part of the solution, the second order and the blended scheme are very similar (the results by the blended scheme is slightly more dissipative than those of the second order unlimited scheme).

The last Figure show that our implementation of the boundary conditions is effective. If we impose strongly the boundary conditions, as in Figure 7–(a) This has to be compared with Figure 5. The Figure 7 show that there is a strong boundary layer on parts of the outer boundary (where the isolines agglomerate). This is not true for the Figure 7–(a). In fact, on some parts of the outer boundary, the compatibility condition of [3] is true, so that one can impose the boundary conditions strongly, and on other parts this is not true and we have to apply them weakly. This partition of the boundary is not known a priori : our implementation take this into account automatically.

9 Conclusions

We have describe several technique for the solution of first order Hamilton Jacobi equations. We have tried to explain the hidden details and the origins of the schemes. Several theoretical results are provided,



(a) (b)

Figure 5: (a) : Solution of problem (33), min$=-3$, max$=-1.53$. (b) : Solution of problem (34), min$=-3$, max$=-1,47$

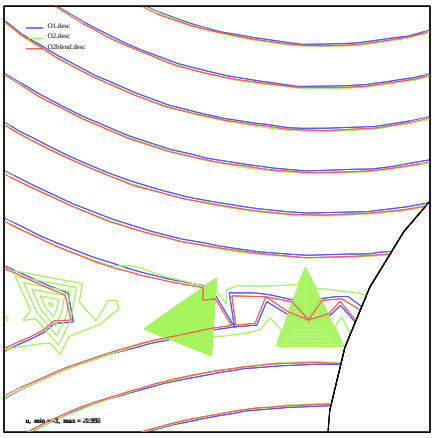
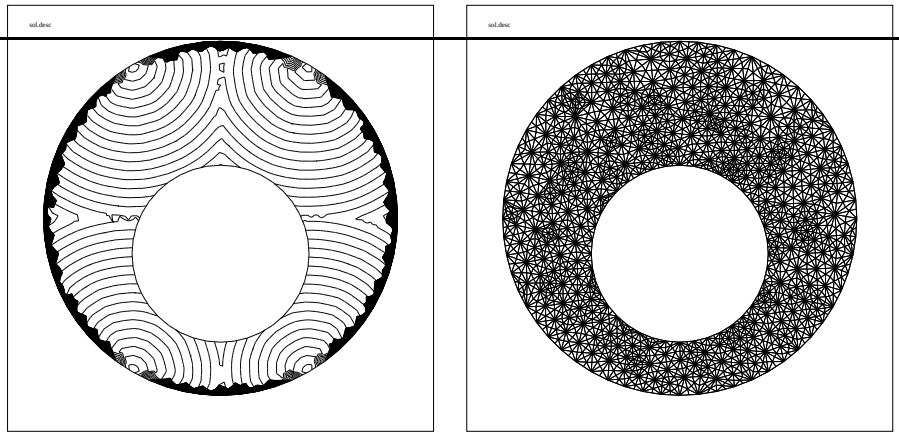


Figure 6: Comparison of the solution of problem (34) with several schemes. In blue: the first order scheme ($\min = -3, \max = -1.993$), in green : second order unlimited scheme ($\min = -3, \max = 23.25$), in red : second order blended scheme ($\min = -3, \max = -1.996$).



(a) (b)

Figure 7: (a) : Solution of problem (34) when the boundary conditions are imposed strongly, (b) : mesh

the proofs are given in the references. Once more, there are many other methods for solving these problems, some are very general, some are specially tuned for a specific example such as computing a distance function which is one of the key ingredient of the level set method.

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